Metronome acceleration

Thomas Orgis

January 3, 2015

Spacetime Analogy

For some reason I keep confusing myself with the units in beat space and the definition of an acceleration in reciprocal time only. How's that with the usual motion over distance s during time t?

The speed is

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \ , \tag{1}$$

while the acceleration is given as speed increase per covered distance,

$$a = \frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \ . \tag{2}$$

This definition allows to compute the speed after a certain covered distance,

$$v(s) = v_0 + \int_{s_0}^{s} a(s')\mathrm{d}s' \tag{3}$$

$$\overset{a=\mathrm{const}}{\Leftrightarrow} v(s) = v_0 + a(s - s_0) \ .$$

That can be again regarded as a differential equation linking space and time increments, to be solved for the time difference corresponding to two points in space,

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = v_0 + a(s - s_0)$$

$$\Leftrightarrow \mathrm{d}t = \frac{1}{v_0 + a(s - s_0)}\mathrm{d}s \tag{4}$$

$$\Leftrightarrow t_1 - t_0 = \int_{s_0}^{s_1} \frac{1}{v_0 + a(s - s_0)}\mathrm{d}s$$

... substituting Equation 3 ...

$$v = v_0 + a(s - s_0) \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}s} = a \Rightarrow \mathrm{d}s = \frac{1}{a}\mathrm{d}v$$

$$\Leftrightarrow t_1 - t_0 = \frac{1}{a}\int_{v(s_0)}^{v(s_1)}\frac{1}{v}\mathrm{d}s .$$

$$(5)$$

Hence, the duration of a certain distance interval in time is given by

$$\Delta t = t_1 - t_0 = \frac{1}{a}\ln\frac{v(s_1)}{v(s_0)} . \tag{6}$$

How should one interpret the dimension of the acceleration? The unit s^{-1} could be seen as (m/s) / m, h^{-1} as (km/h) / km. One mixed variant could be (km/h) / m, which boils down to $1000\,\mathrm{h}^{-1}$.

Real Deal

The analogy can be carried forward, identifying the distance s with the position in the score, counted in bars, beats, whatever you like. A sensible unit might be a whole note, denoted with the unit symbol n. A quarter note equals $\Delta s_{note} = 1/4\,\mathrm{n}$, an eigth $1/8\,\mathrm{n}$. A bar in 6/8 meter has length $\Delta s_{\mathrm{bar}} = 3/4\,\mathrm{n}$, while in 4/4 meter it's an unsurprising $\Delta s_{\mathrm{bar}} = 1\,\mathrm{n}$.

Dominic explained the meaning of the acceleration parameter c as such: After c bars, the tempo increased by $1\,\mathrm{bpm}$. The tempo is always related to quarter notes, so that one bar of three quarters length at constant tempo $v = 120\,\mathrm{bpm}$, be it either in 3/4 or in 6/8 meter, takes $1.5\,\mathrm{s}$ to complete. With the bar length

The acceleration value, based on quarter beats per minute, is given by

$$a \cdot (c\Delta s_{\mathrm{bar}}) = 1\,\mathrm{bpm} = 1\frac{1/4\,\mathrm{n}}{\mathrm{min}} \Leftrightarrow \frac{a}{\mathrm{min}^{-1}} = \frac{1}{c}\frac{1/4\,\mathrm{n}}{\Delta s_{\mathrm{bar}}} . \tag{7}$$

The tempo $v(s)$ at a certain point into the score is given by Equation 3 using this acceleration. Also, a time interval for a certain beat can be computed using Equation 6 and the tempo values at the beginning of the converned beat (s_0) and the beginning of the next one (s_1).

It is evident from Equation 6 that all this is only applicable if there indeed is a non-zero acceleration a and the tempo is always non-zero. The latter point is illustrative in the spacetime analogy with the conventional definition of a constant acceleration per time interval,

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2} , \tag{8}$$

where everything is defined nicely. It *is* a difference whether the acceleration is defined using elapsed time or covered distance. If you start out with $v_0 = 0$ (in whatever units), an acceleration that is defined to increase v with each increment of distance cannot take hold, as there is no infinitisemal distance to work on at the beginning. You can't get going if you're not going. The solution makes sense: In that case, there is no solution.

Negative acceleration is of course allowed, as long as it is non-zero. For constant tempo, the trivial

$$\Delta t = \frac{\Delta s}{v} \tag{9}$$

applies.

Derived Acceleration

Actually, the usual notion inside a score is to change tempo from v_0 to v_1 within a certain number of bars. In place of Equation 7, the acceleration then simply is given by the ratio of the tempo difference and the corresponding distance in the score,

$$a = \frac{v(s_1) - v(s_0)}{s_1 - s_0} = \frac{\Delta v}{\Delta s} \ . \tag{10}$$

This the obvious sensible default to use when a part length in bars is given without acceleration.

Mean Tempo

To compare with Dominic's empirical mean tempo for a beat,

$$\bar{v} = \frac{v_0 - v_1}{\ln \frac{v_0}{v_1}} = -\frac{v_1 - v_0}{\ln \frac{v_0}{v_1}} = \frac{v_1 - v_0}{\ln \frac{v_1}{v_0}} \tag{11}$$

(written to avoid direct logarithms of quantities with units attached), the acceleration in Equation 6 can be replaced by Equation 10. This yields

$$\Delta t = \frac{\Delta s}{\Delta v} \ln \frac{v_1}{v_0} \Leftrightarrow \bar{v} = \frac{\Delta s}{\Delta t} = \frac{\Delta v}{\ln \frac{v_1}{v_0}} \ , \tag{12}$$

which is the same as Dominic's expression.

In Terms of Time

One could also define the metronome tempo acceleration in terms of time. Instead of being given in units of inverse time, it would be beats-per-minute per minute. But how easy is it to solve for the length of a beat Δs then?

$$v(t) = v_0 + a(t - t_0)$$

$$\Delta s = \int_{t_0}^{t_1} v_0 + a(t - t_0)\mathrm{d}t \tag{13}$$

$$= v_0 \Delta t + \frac{a}{2}(\Delta t)^2$$

$$0 = (\Delta t)^2 + \frac{2}{a}v_0\Delta t - \frac{2}{a}\Delta s$$

$$\Delta t = -\frac{v_0}{a} \pm \sqrt{\left(\frac{v_0}{a}\right)^2 + \frac{2}{a}\Delta s} \tag{14}$$

This is straightfoward, allowing for the choice of the positive Δt. Now, what is desired? Constant acceleration in space (beat count) or in time?